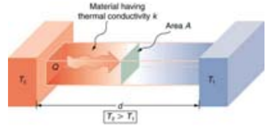
CHAPTER 8

Ordinary Differential Equations

1. INTRODUCTION

- Ordinary differential equation
  - Differential equation
    - An equation containing derivative
  - Partial differential equation
    - is contains partial derivatives
  - Ordinary differential equation
    - Excepts partial derivatives



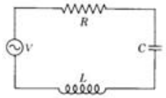


Figure 1.1

- Examples of differential equation
  - $F = ma, F = dv/dt, F = d^2r/dt^2$
  - the rate of change of temperature $\frac{dQ}{dt} = kA\frac{dT}{dx}$
  - A simple series circuit containing a resistance R, a capacitance C, an inductance L, and a source of emf V .

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V.$$

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}$$

- Order of a differential equation
  - the highest derivative in the equation

$$y' + xy^2 = 1,$$
$$xy' + y = e^x,$$
$$\frac{dv}{dt} = -g,$$
$$L\frac{dI}{dt} + RI = V,$$
$$m\frac{d^2r}{dt^2} = -kr$$

  - A linear differential equation (with x as independent and y as dependent variable)

$$a_0y + a_1y' + a_2y'' + a_3y''' + \cdots = b,$$

  - nonlinear equations

$y' = \cot y$ (not linear because of the term $\cot y$);
$yy' = 1$ (not linear because of the product yy');
$y'^2 = xy$ (not linear because of the term y'^2).

A *solution* of a differential equation (in the variables x and y) is a relation between x and y which, if substituted into the differential equation, gives an identity.

Example 1.

(12.9) $$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du,$$

(12.10) $$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha.$$

Example 1. The relation

(1.5) $$y = \sin x + C$$

is a solution of the differential equation

(1.6) $$y' = \cos x$$

because if we substitute (1.5) into (1.6) we get the identity $\cos x = \cos x$.

Example 2. The equation $y'' = y$ has solutions $y = e^x$ or $y = e^{-x}$ or $y = Ae^x + Be^{-x}$ as you can verify by substitution.

- General Solution

> Any *linear* differential equation of order n has a solution containing n independent arbitrary constants, from which *all* solutions of the differential equation can be obtained by letting the constants have particular values. This solution is called the *general* solution of the linear differential equation.

- Particular solution

Example 3.

Example 3. Find the distance which an object falls under gravity in t seconds if it starts from rest.

$$\frac{d^2x}{dt^2} = g.$$

$$\frac{dx}{dt} = gt + \text{const.} = gt + v_0,$$

- General solution

$$x = \tfrac{1}{2}gt^2 + v_0 t + x_0,$$

- Particular solution

$$x = \tfrac{1}{2}gt^2.$$

Example 4.

Example 4. Find the solution of $y'' = y$ which passes through the origin and through the point $(\ln 2, \frac{3}{4})$.

- General solution

$$y = Ae^x + Be^{-x}$$

- If the given points satisfy the equation of the curve, we must have

$$0 = A + B \quad \text{or } A = -B,$$

$$\tfrac{3}{4} = Ae^{\ln 2} + Be^{-\ln 2} = A \cdot 2 + B \cdot \tfrac{1}{2} = 2A - \tfrac{1}{2}A = \tfrac{3}{2}A.$$

$$A = -B = \tfrac{1}{2},$$

- Particular solution

$$y = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

- Conditions for particular solution
  - boundary conditions
    - The given conditions which are to be satisfied by the particular solution
  - initial conditions
    - conditions at $t = 0$
- Slope field
  - direction field, or vector field.
  - a short line (or vector) centered on each point and with the slope y at that point

Example 5. In Figure 1.2, we have plotted a "slope field" for the differential equation $y' = \cos x$. Note how you can trace the general shape of the solution curves, even without knowing from Example 1 that their equations are $y = \sin x + C$.

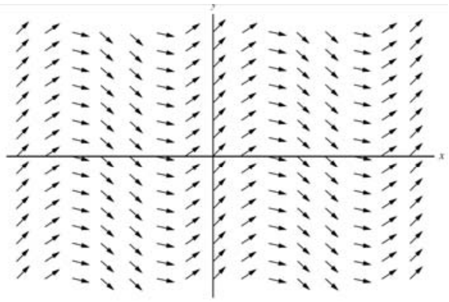
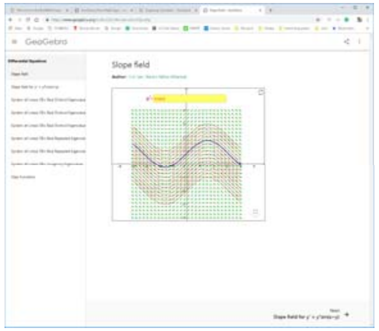


Figure 1.2



8-2. SEPARABLE EQUATIONS

- Evaluate an integral

$$y = \int f(x)\,dx, \tag{2.1}$$

- Solving a differential equation

$$y' = \frac{dy}{dx} = f(x). \tag{2.2}$$

- Separating the variables
  - the equation is separable

$$dy = f(x)\,dx. \tag{2.3}$$

Example 1.

Example 1. The rate at which a radioactive substance decays is proportional to the remaining number of atoms. If there are N_0 atoms at $t = 0$, find the number at time t.

- The proportionality constant λ is called the decay constant

$$\frac{dN}{dt} = -\lambda N.$$

$$N = N_0 e^{-\lambda t}.$$

Example 2.

Example 2. Solve the differential equation

$$xy' = y + 1. \tag{2.6}$$

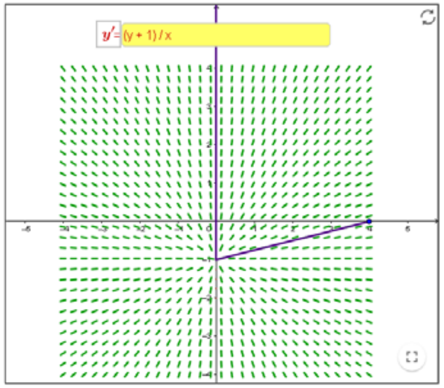
$$\frac{y'}{y+1} = \frac{1}{x} \quad \text{or} \quad \frac{dy}{y+1} = \frac{dx}{x}.$$

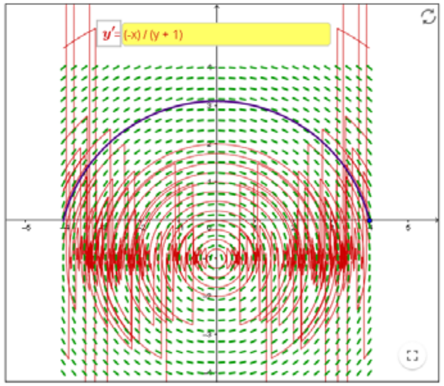
$$\ln(y+1) = \ln x + \text{const.} = \ln x + \ln a = \ln(ax).$$

$$y + 1 = ax.$$

- This general solution represents a family of curves in the (x, y) plane,
- one curve for each value of the constant a
- the general solution (2.9) a family of solutions of the differential equation (2.6)

- Orthogonal Trajectories





https://www.geogebra.org/m/bkA2erJs#material/m3Zgkq9g

Orthogonal Trajectories

- Orthogonal Trajectories

$$y' = a.$$

$$y' = \frac{y+1}{x}.$$

  - Slope of the orthogonal trajectory curve

$$y' = -\frac{x}{y+1}$$

$$(y+1)\,dy = -x\,dx,$$

$$\tfrac{1}{2}y^2 + y = -\tfrac{1}{2}x^2 + C,$$

$$x^2 + y^2 + 2y = 2C,$$

$$x^2 + (y+1)^2 = 2C + 1.$$

  - a family of circles with centers at the point (0, −1).

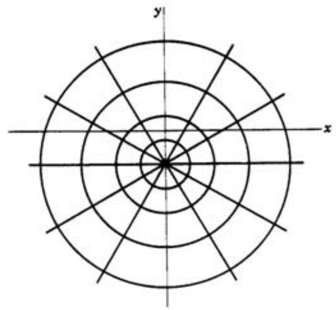


Figure 2.1

Nonlinear Differential Equations

Example 3. Solve the differential equation $y' = \sqrt{1-y^2}$ and computer plot the slope field and a set of solution curves. Find particular solutions satisfying (a) $y = 0$ when $x = 0$, and (b) $y = 1$ when $x = 0$.

We separate variables and integrate to get

$$\frac{dy}{\sqrt{1-y^2}} = dx, \quad \arcsin y = x + \alpha, \quad y = \sin(x+\alpha).$$

- Singular solution
  - $y \equiv 1, y \equiv -1$
- (0, 0)
  - $y = \sin(x)$
- (0, 1)
  - $y = \sin(x + \alpha)$

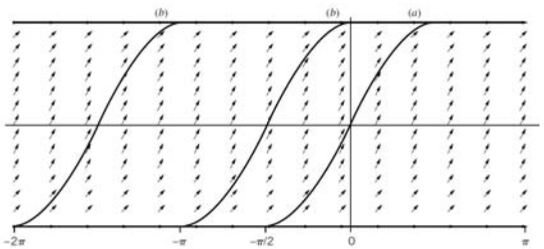


Figure 2.2

8-3 LINEAR FIRST-ORDER EQUATIONS

- First-order equation

$$y' + Py = Q,$$

  - where P and Q are functions of x. Q=0;

$$y' + Py = 0 \quad \text{or} \quad \frac{dy}{dx} = -Py$$ is separable.

$$\frac{dy}{y} = -P\,dx,$$

$$\ln y = -\int P\,dx + C,$$

$$y = e^{-\int P\,dx + C} = Ae^{-\int P\,dx}$$

  - Let us simplify

$$I = \int P\,dx.$$ then $$\frac{dI}{dx} = P$$

- as $y = Ae^{-I}$

$$ye^I = A.$$

- differentiate

$$\frac{d}{dx}(ye^I) = y'e^I + ye^I\frac{dI}{dx} = y'e^I + ye^I P = e^I(y' + Py),$$

$$\frac{d}{dx}(ye^I) = e^I(y' + Py) = Qe^I. \qquad (3.1) \quad y' + Py = Q,$$

$$(3.9) \quad \left.\begin{aligned} ye^I &= \int Qe^I\,dx + c, \qquad \text{or} \\ y &= e^{-I}\int Qe^I\,dx + ce^{-I}, \end{aligned}\right\} \quad \text{where} \quad I = \int P\,dx.$$

Example 1.

Example 1. Solve $(1 + x^2)y' + 6xy = 2x$. In the form of (3.1), this is

$$y' + \frac{6x}{1+x^2}y = \frac{2x}{1+x^2}.$$

$$I = \int \frac{6x}{1+x^2}\,dx = 3\ln(1+x^2)$$

$$e^I = e^{3\ln(1+x^2)} = (1+x^2)^3$$

$$ye^I = \int \frac{2x}{1+x^2}(1+x^2)^3\,dx = \int 2x(1+x^2)^2\,dx = \tfrac{1}{3}(1+x^2)^3 + c$$

$$y = \tfrac{1}{3} + \frac{c}{(1+x^2)^3}.$$

- Computer gives the answer

$$y = \frac{3x^2 + 3x^4 + x^6}{3(1+x^2)^3} + \frac{A}{(1+x^2)^3}.$$

Example 2.

Example 2. Radium decays to radon which decays to polonium. If at $t = 0$, a sample is pure radium, how much radon does it contain at time t?

Let N_0 = number of radium atoms at $t = 0$,
N_1 = number of radium atoms at time t,
N_2 = number of radon atoms at time t,
λ_1 and λ_2 = decay constants for Ra and Rn.

- Equation for radium

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \qquad N_1 = N_0 e^{-\lambda_1 t}.$$

- The rate at which radon is being created is the rate at which radium is decaying,

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2, \qquad \text{or}$$

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1 = \lambda_1 N_0 e^{-\lambda_1 t}.$$

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1 = \lambda_1 N_0 e^{-\lambda_1 t}.$$

$$I = \int \lambda_2\,dt = \lambda_2 t,$$

$$N_2 e^{\lambda_2 t} = \int \lambda_1 N_0 e^{-\lambda_1 t} e^{\lambda_2 t}\,dt + c$$

$$= \lambda_1 N_0 \int e^{(\lambda_2 - \lambda_1)t}\,dt + c = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{(\lambda_2-\lambda_1)t} + c,$$

- if $\lambda_1 \neq \lambda_2$

$$0 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} + c \qquad \text{or} \qquad c = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}.$$

$$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}).$$

8-4. OTHER METHODS FOR FIRST-ORDER EQUATIONS

The Bernoulli Equation

$$y' + Py = Qy^n,$$

- the change of variable

$$z = y^{1-n}.$$

$$z' = (1-n)y^{-n}y'.$$

- Next multiply (4.1) by $(1-n)y^{-n}$

$$\begin{aligned} (1-n)y^{-n}y' + (1-n)Py^{1-n} &= (1-n)Q, \\ z' + (1-n)Pz &= (1-n)Q. \end{aligned}$$

Example 2.

Example 2. We can use (12.17) to evaluate a definite integral. Using $f(x)$ in Figure 12.1, we find

$$\int_0^\infty \frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1, \quad \frac{\pi}{4} \text{ for } |x| = 1, \\ 0 & \text{for } |x| > 1. \end{cases} \tag{12.18}$$

- Fourier integral represents the midpoint of the jump in f(x) at |x| = 1. If we let x = 0,

$$\int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha = \frac{\pi}{2}. \tag{12.19}$$